AN EFFICIENT FINITE ELEMENT APPROACH FOR THE ANALYSIS OF THREE-DIMENSIONAL TRANSMISSION LINE DISCONTINUITIES USING AN ASYMPTOTIC BOUNDARY CONDITION

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Abstract In this paper, a novel technique is developed for efficient finite element solution of open region three-dimensional transmission line structures in the quasi-TEM regime. Starting with the general form of the solution to the three-dimensional Laplace's equation in spherical coordinates, a set of asymptotic boundary condition (ABC) operators is derived. The second-order ABC is then applied on a conformable outer boundary for the purpose of truncating the FEM mesh in an efficient manner. To illustrate its application, the method is used to compute the capacitance of a rectangular microstrip patch and the results are found to be in good agreement with data published elsewhere.

I. Introduction

Typically, a printed circuit board contains not only uniform transmission line etches that are essentially invariant in the longitudinal direction, but also chip sockets and connectors for interboard communication that can not be modeled as uniform lines. Furthermore, the transmission lines themselves may have various discontinuities such as bends, changes in width, open circuits, gaps and steps. In recent years, there has been an increasing interest in modeling such discontinuities, and a number of papers [1-7] have been written on this subject. In most of these papers, the integral equation technique has been used to study planar conductors and structures containing a homogeneous dielectric with planar interfaces. Castillo [8] has used the finite element method (FEM), which can handle any arbitrary configuration of conductors and dielectrics. When using the FEM, one needs to deal with the practical problem of mesh truncation and the large number of mesh nodes. Similar to the two-dimensional problems, the most widely-used approach for dealing with the mesh truncation problem for the three-dimensional geometry is to place a fictitious, box-type conducting enclosure sufficiently far from the structure [8]. This approach, which assumes that the field decays significantly before reaching the outer boundary, typically results in an undesirable large mesh, especially for three-dimensional geometries. Previously, the authors have introduced an asymptotic boundary condition (ABC), which provided them with an efficient means for dealing with open region two-dimensional microwave transmission line problems in the quasi-static regime [10]. The usefulness of the ABC to obtain an accurate solution to a problem with a reasonable number of node points was demonstrated in that paper. For three-dimensional problems, where the total number of mesh points is usually large, it is expected that the availability of an accurate ABC will play an even more crucial role in the solution of practical problems.

In this paper, we derive an asymptotic boundary condition for three-dimensional open region problems in the quasi-static regime. This asymptotic boundary condition enables us to bring the outer boundary much closer to the structure than would be possible with the p.e.c. artificial boundary. In order to reduce the number of unknowns as much as possible, we have chosen an outer boundary in the shape of a box because it is the most conformable to the structures considered.

II. Derivation of the Three-Dimensional Asymptotic Boundary Conditions

The asymptotic or the absorbing boundary condition has seen an increasing use in connection with the partial differential equation (PDE) techniques for solving open region electromagnetic problems because it preserves the sparsity of the discretized PDE matrix [9-10]. In this section, we derive an asymptotic boundary condition for three-dimensional quasi-static problems.

Consider the three-dimensional open region

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$$\xi_1(x,y,z) = -\frac{zxy}{r^2} \quad (17)$$

where $\rho = (x^2 + y^2)^{1/2}$ and $r = (x^2 + y^2 + z^2)^{1/2}$. Similar expressions can be obtained for the faces where y=constant and z=constant.

It is evident that one needs to use the appropriate normal derivative expression on different faces of the box-shaped outer boundary. Obviously, it is much easier to choose a spherical outer boundary where the normal derivative is simply u_r. However, for the purpose of truncating the unbounded region surrounding the transmission lines in an efficient manner, one needs to use a conformable outer boundary which, as mentioned earlier, is typically a box-shaped surface for three-dimensional transmission line structures. The asymptotic boundary condition expressions that we have just derived will be implemented in the finite element scheme in the next section.

III. Finite Element implementation of the Asymptotic Boundary Condition

As indicated earlier, the region of interest, Ω_T, is bounded by an artificial boundary, Γ_2, to limit the number of unknowns. Over the bounded region, the Laplace equation is solved at a finite number of grid points. This equation is discretized through the use of a weak form of variational representation.

Multiplying the Laplace equation (1) by a testing function f and integrating over the volume Ω_T, we get

$$\int_{\Omega_T} f\nabla\cdot(\varepsilon\nabla u)\, dv = 0 \quad (18)$$

Using the Green's second identity, we can rewrite (18) as

$$\int_{\Omega_T} \varepsilon\nabla u\cdot\nabla f\, dv = \int_{\Gamma_2} f\varepsilon \frac{\partial u}{\partial n}\, ds \quad (19)$$

Next, we discretize the region Ω_T into tetrahedral elements. The triangular faces of the outermost elements make up the outer boundary Γ_2. In the Finite Element formulation, the implementation of (19) is carried out on an element-by-element basis. For all but the outermost tetrahedral elements, the right-hand side of equation (19) is zero. The asymptotic boundary condition is needed to treat those outermost elements.

For those elements having a face on the surface prescribed by x=constant, where the outward normal is in the plus or minus x-direction, the asymptotic boundary condition given in (11) may be incorporated into (19) to yield

$$\begin{aligned}\int_{\Omega_T} \varepsilon\nabla u\cdot\nabla f\, dv = \pm\int_{\Gamma_2} f\varepsilon\{\, &\alpha_1(x,y,z)u \\ &+ \beta_1(x,y,z)u_z + \gamma_1(x,y,z)u_{zz} \\ &+ \zeta_1(x,y,z)u_y + \eta_1(x,y,z)u_{yy} \\ &+ \xi_1(x,y,z)u_{yz}\}\, dydz\end{aligned} \quad (20)$$

Similar expressions can be obtained for the elements having a face on the surface prescribed by y=constant and z=constant.

IV. Numerical Results

A rectangular section of microstrip transmission line of length L, width W, and height H above the ground plane is shown in Figure 2. The outer boundary Γ_2 was chosen to have the shape of a box. Using the same mesh, we solved the potential problem twice, first by applying the asymptotic boundary condition on the outer boundary, and second by placing a perfect electric conducting shield at the same location. After solving for the electrostatic potential, we computed the normalized capacitance CH/ε(area) for both cases. Tables 1 and 2 show the results of computation for the normalized capacitance for different values of L/W and for three dielectric constants (ε_r = 1.0, 6.0, 9.6). As Tables 1 and 2 indicate, the asymptotic boundary condition yields more accurate results than those obtainable with a perfectly conducting shield [2]. Clearly, for this problem there is distinct advantage in using an asymptotic boundary condition in place of a p.e.c. shield.

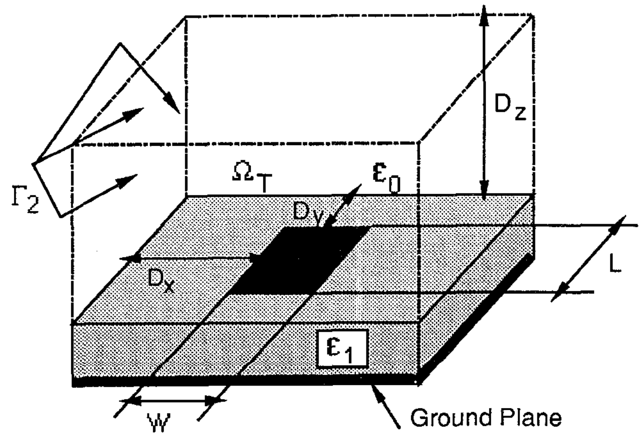

Figure 2. A rectangular microstrip patch enclosed by a box in order to minimize the number of mesh points.

problem consisting of an arbitrarily- shaped discontinuity embedded in a multilayered medium above a ground plane shown in Figure 1. Let Ω_T be the region exterior to the conductors and Γ_2 be the outer boundary. Our objective is to derive an operator which, when applied on the outer boundary, makes the field emulate the asymptotic behavior at infinity, and thus yields an accurate result for the interior region with only a moderate number of nodes. Equivalently, we accomplish this task by imposing an asymptotic boundary condition (ABC) on the field on the outer boundary.

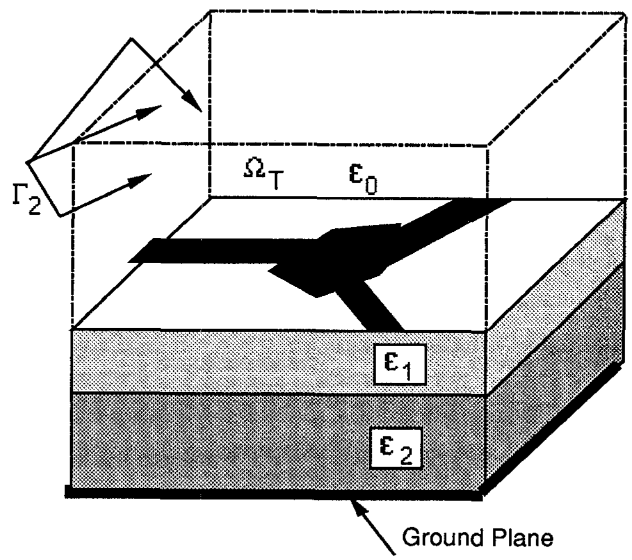

Figure 1. Geometry of a general transmission line discontinuity in a multi-layered dielectric region above a ground plane.

The boundary value problem to be solved can be expressed by the set of equations:

$$\nabla\cdot(\varepsilon\nabla u) = 0 \quad \text{in } \Omega_T \tag{1}$$

$$u = g_i \text{ on the } i^{th} \text{ conductor} \tag{2}$$

$$B_m u = 0 \text{ on } \Gamma_2 \tag{3}$$

where u is the electrostatic potential, g_i is the potential on the conductors, and B_m is the m^{th} order asymptotic boundary operator.

For large r, the general solution of the Laplace equation in spherical coordinates can be written in terms of spherical harmonics and inverse powers of r as

$$u(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-1}^{1} \frac{B_{lm}}{r^{l+1}} Y_{lm}(\theta,\phi) \tag{4}$$

where $Y_{lm}(\theta,\phi)$ are spherical harmonics.

Manipulating (4), it can be shown that an m^{th} order asymptotic boundary condition operator can be written as

$$B_m u = \prod_{j=1}^{m}\left(\frac{\partial}{\partial r} + \frac{2j-1}{r}\right)u \tag{5}$$

As will be shown later, the boundary contribution in the finite element formulation enters into a surface integral representation over the outer boundary, Γ_2, where the integrand is the product of a testing function and the normal derivative of u. As a consequence, the asymptotic boundary condition needs to be imposed on the normal derivative of u. For a spherical outer boundary, the normal derivative is simply the radial one. Using B_2 in conjunction with Laplace's equation in the spherical coordinates, we obtain the following asymptotic boundary condition operator

$$u_r = \alpha(r)u + \beta(r)u_\theta + \gamma(r)u_{\theta\theta} + \xi(r)u_{\phi\phi} \tag{6}$$

where

$$\alpha(r) = -\frac{1}{r} \tag{7}$$

$$\beta(r) = \frac{\cot\theta}{2r} \tag{8}$$

$$\gamma(r) = \frac{1}{2r} \tag{9}$$

$$\xi(r) = \frac{1}{2r\sin^2\theta} \tag{10}$$

As indicated earlier, it is highly desirable to use a conformable outer boundary e.g., a box-shaped surface for three-dimensional transmission line structures, in order to minimize the number of node points as much as possible. We present below the expressions for the appropriate normal derivative for the different faces of the box representing such an outer boundary.

For the x=constant face of the box, the normal derivative is plus or minus u_x. Using the Chain rule, and the relations between the angular and the tangential derivatives, we can express u_x as

$$u_x = \alpha_1(x,y,z)u + \beta_1(x,y,z)u_z + \gamma_1(x,y,z)u_{zz} + \zeta_1(x,y,z)u_y + \eta_1(x,y,z)u_{yy} + \xi_1(x,y,z)u_{yz} \tag{11}$$

where

$$\alpha_1(x,y,z) = -\frac{x}{r^2} \tag{12}$$

$$\beta_1(x,y,z) = -z\frac{3x^3+4xy^2}{2r^2\rho^2} \tag{13}$$

$$\gamma_1(x,y,z) = \frac{x^3+xy^2}{2r^2} \tag{14}$$

$$\zeta_1(x,y,z) = \frac{4x^3yz^2+3xy^3z^2-xy(\rho^4+2r^2\rho^2)}{2r^2\rho^4} \tag{15}$$

$$\eta_1(x,y,z) = \frac{z^2y^2x+x^3r^2}{2r^2\rho^2} \tag{16}$$

Table 1. Normalized capacitance $\left(\frac{CH}{\varepsilon \cdot \text{Area}}\right)$ for $\frac{L}{W}$=0.2, $\frac{H}{W}$=0.2, D_x=D_y=D_z=0.5.

ε_r	P.E.C. Shield	ABC (Present Method)	Reference [6]
1.0	1.34	3.73	3.5
6.0	1.04	2.25	2.2
9.6	1.02	2.12	2.1

Table 1. Normalized capacitance $\left(\frac{CH}{\varepsilon \cdot \text{Area}}\right)$ for $\frac{L}{W}$=1.0, $\frac{H}{W}$=1.0, D_x=D_y=D_z=2.0.

ε_r	P.E.C. Shield	ABC (Present Method)	Reference [6]
1.0	2.19	4.90	5.0
6.0	1.36	3.11	3.4
9.6	1.31	2.84	2.9

V. Conclusions

Starting from the general solution of the Laplace equation in spherical coordinates, we derived a set of asymptotic boundary conditions for three-dimensional quasi-static problems for a spherical outer boundary. The second-order boundary condition was then generalized to a box-shaped outer boundary, for the purpose of truncating the mesh in an efficient manner, and implemented in the finite element method to solve the potential problem of a rectangular microstrip patch. The numerical results show that the asymptotic boundary conditions yield more accurate results than those obtainable with a perfectly conducting shield placed at the same location.

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